Calculate the limit
https://www.linkedin.com/groups/8313943/8313943-6415759172310228995
lim n → ∞ ((n + 1)(n + 2) · · · (3n)/n^(2n))^1/n.
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Solution 1.
Let $a_n := \dfrac{(n+1)(n+2)\ldots(3n)}{n^{2n}}, n \in \mathbb{N}$. Since $\dfrac{a_{n+1}}{a_n} =$
$\dfrac{(n+2)(n+2)\ldots(3n)(3n+1)(3n+2)(3n+3)}{(n+1)^{2(n+1)}} \cdot \dfrac{n^{2n}}{(n+1)(n+2)\ldots(3n)} = \dfrac{3n^{2n}(3n+1)(3n+2)}{(n+1)^{2(n+1)}}$
$\dfrac{3(3+1/n)(3+2/n)}{(1+1/n)^{2n}}$ then $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{27}{e^2}$ and by* GM Limit Theorem (Geometric Mean)
$\lim\limits_{n\to\infty}\left(\dfrac{(n+1)(n+2)\ldots(3n)}{n^{2n}}\right)^{1/n} = \lim\limits_{n\to\infty}\sqrt[n]{a_n} = \dfrac{27}{e^2}$.
Solution 2.
Noting that $\dfrac{(n+1)(n+2)\ldots(3n)}{n^{2n}} = \dfrac{(3n)!}{(3n)^{3n}} \cdot \dfrac{n^n}{n!} \cdot 3^{3n}$ and using well known**
$\lim\limits_{n\to\infty}\left(\dfrac{n!}{n}\right)^{1/n} = \dfrac{1}{e}$
we obtain that $\lim\limits_{n\to\infty}\left(\dfrac{(3n)!}{(3n)^{3n}}\right)^{1/3n} = \dfrac{1}{e}$ (as limit of subsequence) and, therefore,
$\lim\limits_{n\to\infty}\left(\dfrac{(n+1)(n+2)\ldots(3n)}{n^{2n}}\right)^{1/n} = 3^3 \lim\limits_{n\to\infty}\left(\dfrac{(3n)!}{(3n)^{3n}}\right)^{1/n} \cdot \lim\limits_{n\to\infty}\left(\dfrac{n}{n!}\right)^{1/n} = 27 \cdot \dfrac{1}{e^3} \cdot e = \dfrac{27}{e^2}$.
* It is the short name of Cauchy's Second Limit Theorem (and it is also can be considered as
particular case of Stolz-Cezaro Theorem in multiplicative form):
Let $\lim\limits_{n\to\infty} a_n = a > 0$. Then $\lim\limits_{n\to\infty}\sqrt[n]{a_1 a_2 \ldots a_n} = a$
and here we apply it in the form:
Let $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = a$. Then $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = a$.
** Using double inequality $\left(\dfrac{n}{e}\right)^n < n! < \dfrac{(n+1)^{n+1}}{e^n}$ (which can be easily proved by Math Induction) we obtain $\dfrac{1}{e^n} < \dfrac{n!}{n^n} < \dfrac{n+1}{e^n} \cdot \left(1+\dfrac{1}{n}\right)^n \Leftrightarrow$
$\dfrac{1}{e} < \dfrac{\sqrt[n]{n!}}{n} < \dfrac{1}{e}\sqrt[n]{n+1} \cdot \left(1+\dfrac{1}{n}\right)$ and since $\lim\limits_{n\to\infty}\sqrt[n]{n+1}\left(1+\dfrac{1}{n}\right) = 1$ then $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{n}$.